Exercise 1J

1 Expand the following up to the term in x^3, given that $|x| < \frac{1}{2}$.

a $\frac{1}{(1+x)}$ b $\frac{1}{(1-2x)^2}$

c $\frac{2}{(1+2x)}$ d $\frac{2}{(1-x)^3}$

(a) $\frac{1}{(1+x)}$

$= (1+x)^{-1}$

$= 1 + (-1)(x) + \frac{(-1)(-2)(x^2)}{2!} + \frac{(-1)(-2)(-3)(x^3)}{3!} + \dots$

$= 1 - x + x^2 - x^3 + \dots$

(b) $\frac{1}{(1-2x)^2}$

$= (1-2x)^{-2}$

$= 1 + (-2)(-2x) + \frac{(-2)(-3)(-2x^2)}{2!} + \frac{(-2)(-3)(-4)(-2x^3)}{3!} + \dots$

$= 1 + 4x + 12x^2 + 32x^3 + \dots$

(c) $\frac{2}{(1+2x)}$

$= 2(1+2x)^{-1}$

$= 2(1) + 2(-1)(2x) + \frac{2(-1)(-2)(2x^2)}{2!} + \frac{2(-1)(-2)(-3)(2x^3)}{3!} + \dots$

$= 2 - 4x + 8x^2 - 16x^3 + \dots$

(d) $\frac{2}{(1-x)^3}$

$= 2(1-x)^{-3}$

$= 2(1) + 2(-3)(-x) + \frac{2(-3)(-4)(-x)^2}{2!} + \frac{2(-3)(-4)(-5)(-x)^3}{3!} + \dots$

$= 2 + 6x + 12x^2 + 20x^3 + \dots$

2 Find the first four terms of each of the following expansions where $|x| < \frac{1}{10}$:

a $\sqrt{(1+2x)}$ **b** $(1+x)^{\frac{3}{2}}$

c $(1-3x)^{-\frac{1}{2}}$ **d** $2(1+x)^{\frac{1}{3}}$

(a) $\sqrt{(1+2x)}$

$= (1+2x)^{\frac{1}{2}}$

$= 1 + (\frac{1}{2})(2x) + \frac{(\frac{1}{2})(-\frac{1}{2})(2x^2)}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(2x^3)}{3!} + \ldots$

$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots$

(b) $(1+x)^{\frac{3}{2}}$

$= 1 + (\frac{3}{2})(x) + \frac{(\frac{3}{2})(\frac{1}{2})(x^2)}{2!} + \frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(x^3)}{3!} + \ldots$

$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{x^3}{16} + \ldots$

(c) $(1-3x)^{-\frac{1}{2}}$

$= 1 + (-\frac{1}{2})(-3x) + \frac{(-\frac{1}{2})(-\frac{3}{2})(-3x)^2}{2!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-3x)^3}{3!} + \ldots$

$= 1 + \frac{3}{2}x + \frac{27}{8}x^2 + \frac{135}{16}x^3 + \ldots$

(d) $2(1+x)^{\frac{1}{3}}$

$= 2(1) + 2(\frac{1}{3})(x) + \frac{2(\frac{1}{3})(-\frac{2}{3})(x^2)}{2!} + \frac{2(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(x^3)}{3!} + \ldots$

$= 2 + \frac{2}{3}x - \frac{2}{9}x^2 + \frac{10}{81}x^3 + \ldots$

3 Show that $\sqrt{\dfrac{1-x}{1+x}} \simeq 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{2}$, where $|x| < 1$.

$$(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$= \left[1 + (\tfrac{1}{2})(-x) + \frac{\frac{1}{2}(-\frac{1}{2})(-x)^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-x)^3}{3!} + \dots\right] \times \left[1 + (-\tfrac{1}{2})(x) + \frac{(-\frac{1}{2})(-\frac{3}{2})(x^2)}{2!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(x^3)}{3!}\right]$$

$$= \left[1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \dots\right] \times \left[1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \dots\right]$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \dots$$

$$- \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{3}{16}x^3 + \dots$$

$$- \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \dots$$

$$+ \qquad - \tfrac{1}{16}x^3 + \dots$$

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \dots$$

Shown

4 Show that

$\dfrac{x}{(1+x)^2} \simeq x - 2x^2 + 3x^3 - 4x^4 + \dots, |x| < 1.$

$$x(1+x)^{-2}$$

$$= x(1) + x(-2)(x) + \frac{x(-2)(-3)(x^2)}{2!} + \frac{x(-2)(-3)(-4)(x^3)}{3!} + \dots$$

$$= x - 2x^2 + 3x^3 - 4x^4 + \dots$$

5 Find the first four terms of the binomial expansion of $(2-3x)^{-3}$, $|x| < \frac{2}{3}$.

$$(2-3x)^{-3} = \left[\frac{1}{\frac{1}{2}(1-\frac{3}{2}x)}\right]^{3}$$

$$= \frac{1}{8}(1-\tfrac{3}{2}x)^{-3}$$

$$= \frac{1}{8}(1) + \frac{1}{8}(-3)(-\tfrac{3}{2}x) + \frac{\frac{1}{8}(-3)(-4)(-\frac{3}{2}x)^2}{2!} + \frac{\frac{1}{8}(-3)(-4)(-5)(-\frac{3}{2}x)^3}{3!} + \dots$$

$$= \frac{1}{8} + \frac{9}{16}x + \frac{27}{16}x^2 + \frac{135}{32}x^3 + \dots$$

6 a Find the first four terms of the binomial expansion of $\sqrt{1-4x}$, $|x| < \frac{1}{4}$.

b Show that the exact value of $\sqrt{1-4x}$ when $x = \frac{1}{100}$ is $\frac{2\sqrt{6}}{5}$.

c Hence, determine $\sqrt{6}$ to 5 decimal places.

(a) $(1-4x)^{\frac{1}{2}}$

$$= 1 + (\tfrac{1}{2})(-4x) + \frac{(\frac{1}{2})(-\frac{1}{2})(-4x)^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-4x)^3}{3!} + \dots$$

$$= 1 - 2x - 2x^2 - 4x^3 + \dots$$

(b) $\sqrt{1-4x}$

$$= \sqrt{1-4(\tfrac{1}{100})}$$

$$= \frac{\sqrt{24}}{\sqrt{25}}$$

$$= \frac{\sqrt{4} \times \sqrt{3\times 2}}{\sqrt{25}}$$

$$= \frac{2\sqrt{6}}{5}$$

(c) $\frac{2\sqrt{6}}{5} = \sqrt{1-\frac{4}{100}}$

$$\sqrt{6} = \frac{5}{2}\left[1 - 2(\tfrac{1}{100}) - 2(\tfrac{1}{100})^2 - 4(\tfrac{1}{100})^3 + \dots\right]$$

$$= 2.44949$$

7 **a** Find the first three terms of the binomial expansion of $\dfrac{1}{\sqrt{1-2x}}$ where $|x| < \dfrac{1}{2}$.

b Hence or otherwise, obtain the expansion of $\dfrac{(2+3x)^3}{\sqrt{1-2x}}$, $|x| < \dfrac{1}{2}$ up to and including the term in x^3.

(a) $(1-2x)^{-\frac{1}{2}}$

$$= 1 + \left(-\tfrac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^2}{2!} + \ldots$$

$$= 1 + x + \frac{3}{2}x^2 + \ldots$$

(b) $(2+3x)^3(1-2x)^{-\frac{1}{2}}$

$$= \left[8 + 36x + 54x^2 + 27x^3\right] \times \left[1 + \left(-\tfrac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-2x)^3}{3!} + \ldots\right]$$

$$= \left[8 + 36x + 54x^2 + 27x^3\right] \times \left[1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \ldots\right]$$

$$= 8 + 8x + 12x^2 + 20x^3 + 36x + 36x^2 + 54x^2 + 54x^3 + 54x^3 + 27x^3 + \ldots$$

$$= 8 + 44x + 102x^2 + 155x^3 + \ldots$$