1 Write the first four terms in the binomial expansion of:

a $\left(1-\frac{x}{3}\right)^{11}$ **b** $\left(1+\frac{x}{2}\right)^{7}$ **c** $\left(x+\frac{2}{x}\right)^{8}$

(a) $\left(1-\frac{x}{3}\right)^{11}$

$= {}^{11}C_0(1)^{11}\left(-\frac{x}{3}\right)^0 + {}^{11}C_1(1)^{10}\left(-\frac{x}{3}\right)^1 + {}^{11}C_2(1)^9\left(-\frac{x}{3}\right)^2 + {}^{11}C_3(1)^8\left(-\frac{x}{3}\right)^3 + \dots$

$= 1 - \frac{11x}{3} + \frac{55x^2}{9} - \frac{55x^3}{9} + \dots$

(b) $\left(1+\frac{x}{2}\right)^{7}$

$= {}^{7}C_0(1)^{7}\left(\frac{x}{2}\right)^0 + {}^{7}C_1(1)^{6}\left(\frac{x}{2}\right)^1 + {}^{7}C_2(1)^5\left(\frac{x}{2}\right)^2 + {}^{7}C_3(1)^4\left(\frac{x}{2}\right)^3 + \dots$

$= 1 + \frac{7x}{2} + \frac{21x^2}{4} + \frac{35x^3}{8} + \dots$

(c) $\left(x+\frac{2}{x}\right)^{8}$

$= {}^{8}C_0(x)^{8}\left(\frac{2}{x}\right)^0 + {}^{8}C_1(x)^{7}\left(\frac{2}{x}\right)^1 + {}^{8}C_2(x)^6\left(\frac{2}{x}\right)^2 + {}^{8}C_3(x)^5\left(\frac{2}{x}\right)^3 + \dots$

$= x^8 + 16x^6 + 112x^4 + 448x^2 + \dots$

2 In each of the following binomial expressions, write down the required term.

a fifth term of $(a - 2b)^{10}$

b third term of $\left(a + \dfrac{4}{a^2}\right)^{11}$

c fourth term of $\left(x - \dfrac{2y}{x}\right)^{8}$

(a) $^{10}C_4(a^{10-5})(-2b)^4$

$= 210a^5(16b^4)$

$= 3360a^5b^4$

(b) $\left(a + \frac{4}{a^2}\right)^{11}$

$= {}^{11}C_2(a)^{11-2}\left(\frac{4}{a^2}\right)^2$

$= 55(a^9)\left(\frac{16}{a^4}\right)$

$= 880a^5$

(c) $\left(x - \frac{2y}{x}\right)^8$

$= {}^{8}C_3(x)^{8-3}\left(-\frac{2y}{x}\right)^3$

$= 56(x^5)\left(-\frac{8y^3}{x^3}\right)$

$= -448x^2y^3$

3 Find the term independent of x in the expansion of $\left(x - \dfrac{2}{x^2}\right)^{12}$.

$^{12}C_r(x)^{12-r}\left(-\frac{2}{x^2}\right)^r = Nx^0$

$x^{12-r}(x^{-2r}) = x^0$

$12 - r - 2r = 0$

$12 = 3r$

$r = 4$

$^{12}C_4(x)^{12-4}\left(-\frac{2}{x^2}\right)^4$

$= 495(x^8)\left(\frac{16}{x^8}\right)$

$= 7920$

4 Use the binomial theorem to expand $\left(2-\frac{x}{5}\right)^4$. Hence find the value of $(1.99)^4$ correct to 5 decimal places.

$\left(2-\frac{x}{5}\right)^4$

$= {}^4C_0(2)^4\left(-\frac{x}{5}\right)^0 + {}^4C_1(2)^3\left(-\frac{x}{5}\right)^1 + {}^4C_2(2)^2\left(-\frac{x}{5}\right)^2 + {}^4C_3(2)^1\left(-\frac{x}{5}\right)^3 + {}^4C_4(2)^0\left(-\frac{x}{5}\right)^4$

$= 16 - \frac{32x}{5} + \frac{24x^2}{25} - \frac{8x^3}{125} + \frac{x^4}{625}$

$(1.99)^4 = \left(2 - \frac{0.05}{5}\right)^4$

$16 - \frac{32x}{5} + \frac{24x^2}{25} - \frac{8x^3}{125} + \frac{x^4}{625}$

$= 15.68239$

5 Find the term in x^6 in the expansion of $\left(x^2-\frac{1}{x}\right)^6$.

${}^6C_r(x^2)^{6-r}\left(-\frac{1}{x}\right)^r = Nx^6$

$x^{12-2r}(x^{-r}) = x^6$

$12-2r-r = 6$

$6 = 3r$

$r = 2$

${}^6C_2(x^2)^{6-2}\left(-\frac{1}{x}\right)^2$

$= 15(x^8)(x^{-2})$

$= 15x^6$

6 **a** Expand $\left(x+\frac{y}{x}\right)^5$.

b Find the coefficient of x^3y^2 in the expansion of $(2x+y)\left(x+\frac{y}{x}\right)^5$.

(a) $\left(x+\frac{y}{x}\right)^5$

$= {}^5C_0(x)^5\left(\frac{y}{x}\right)^0 + {}^5C_1(x)^4\left(\frac{y}{x}\right)^1 + {}^5C_2(x)^3\left(\frac{y}{x}\right)^2 + {}^5C_3(x)^2\left(\frac{y}{x}\right)^3 + {}^5C_4(x)^1\left(\frac{y}{x}\right)^4 + {}^5C_5(x)^0\left(\frac{y}{x}\right)^5$

$= x^5 + 5x^3y + 10xy^2 + 10\frac{y^3}{x} + \frac{5y^4}{x^3} + \frac{y^5}{x^5}$

(b) $(2x+y)\left[{}^5C_0(x)^5\left(\frac{y}{x}\right)^0 + {}^5C_1(x)^4\left(\frac{y}{x}\right)^1 + {}^5C_2(x)^3\left(\frac{y}{x}\right)^2 + {}^5C_3(x)^2\left(\frac{y}{x}\right)^3 + {}^5C_4(x)^1\left(\frac{y}{x}\right)^4 + {}^5C_5(x)^0\left(\frac{y}{x}\right)^5\right]$

$= (2x+y)\left[x^5 + 5x^3y + 10xy^2 + \frac{10y^3}{x} + \frac{5y^4}{x^3} + \frac{y^5}{x^5}\right]$

$y(5x^3y)$

$= 5x^3y^2$ $\therefore$ 5 is the coefficient of x^3y^2

7 Write in factorial notation:

a the coefficient of x^4 in the expansion of $(1+x)^{n+1}$

b the coefficient of x^3 in the expansion of $(1+2x)^n$.

c Find n, given that these two coefficients are equal.

(a) ${}^{n+1}C_4(1)^{n+1-4}(x)^4 = Nx^4$

$\frac{(n+1)!}{(n+1-4)!4!} = N$

$N = \frac{(n+1)!}{(n-3)!4!}$

(b) $^{n}C_3(1)^{n-3}(2x)^3 = Nx^3$

$$\frac{8n!}{(n-3)!3!} = N$$

$$N = \frac{4n!}{3(n-3)!}$$

(c) $\dfrac{(n+1)!}{(n-3)!4!} = \dfrac{4n!}{3(n-3)!}$

$$\frac{(n+1)\cancel{n!}\cancel{(n-3)!}}{\cancel{n!}\cancel{(n-3)!}} = \frac{4!(4)}{3}$$

$$n+1 = 32$$

$$n = 31$$

8 **a** Express $\left(\sqrt{3}-\sqrt{2}\right)^5$ in the form of $a\sqrt{3}+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

b Express $\left(\sqrt{2}-\dfrac{1}{\sqrt{5}}\right)^4$ in the form $a+b\sqrt{10}$, $a, b \in \mathbb{Q}$.

c Express $\left(1+\sqrt{5}\right)^7-\left(1-\sqrt{5}\right)^7$ in the form $a\sqrt{5}$, $a \in \mathbb{Z}$.

(a) $(\sqrt{3}-\sqrt{2})^5$

$= {}^5C_0(\sqrt{3})^5(-\sqrt{2})^0 + {}^5C_1(\sqrt{3})^4(-\sqrt{2})^1 + {}^5C_2(\sqrt{3})^3(-\sqrt{2})^2 + {}^5C_3(\sqrt{3})^2(-\sqrt{2})^3 + {}^5C_4(\sqrt{3})^1(-\sqrt{2})^4 + {}^5C_5(\sqrt{3})^0(-\sqrt{2})^5$

$= (\sqrt{3})^5 + 45(-\sqrt{2}) + 20(\sqrt{3})^3 + 30(-\sqrt{2})^3 + 20(\sqrt{3}) + (-\sqrt{2})^5$

$= 9\sqrt{3} - 45\sqrt{2} + 60\sqrt{3} - 60\sqrt{2} + 20\sqrt{3} - 4\sqrt{2}$

$= 89\sqrt{3} - 109\sqrt{2}$

(b) $\left(\sqrt{2}-\frac{1}{\sqrt{5}}\right)^4$

$= {}^4C_0(\sqrt{2})^4\left(-\frac{1}{\sqrt{5}}\right)^0 + {}^4C_1(\sqrt{2})^3\left(-\frac{1}{\sqrt{5}}\right)^1 + {}^4C_2(\sqrt{2})^2\left(-\frac{1}{\sqrt{5}}\right)^2 + {}^4C_3(\sqrt{2})^1\left(-\frac{1}{\sqrt{5}}\right)^3 + {}^4C_4(\sqrt{2})^0\left(-\frac{1}{\sqrt{5}}\right)^4$

$= 4 + 8\sqrt{2}\left(-\frac{1}{\sqrt{5}}\right) + 12\left(-\frac{1}{\sqrt{5}}\right)^2 + 4\sqrt{2}\left(-\frac{1}{\sqrt{5}}\right)^3 + \left(-\frac{1}{\sqrt{5}}\right)^4$

$= 4 - \frac{8\sqrt{2}}{\sqrt{5}} + \frac{12}{5} - \frac{4\sqrt{2}}{5\sqrt{5}} + \frac{1}{25}$

$= \frac{161}{25} - \frac{44}{5}\left(\frac{\sqrt{2}}{\sqrt{5}}\right)$

$= \frac{161}{25} - \frac{44}{5}\left(\frac{\sqrt{2}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}\right)$

$= \frac{161}{25} - \frac{44\sqrt{10}}{25}$

(c) $(1+\sqrt{5})^7-(1-\sqrt{5})^7$

$(1+\sqrt{5})^7$

$= 1 + 7(\sqrt{5}) + 21(\sqrt{5})^2 + 35(\sqrt{5})^3 + 35(\sqrt{5})^4 + 21(\sqrt{5})^5 + 7(\sqrt{5})^6 + (\sqrt{5})^7$

$= 1 + 7\sqrt{5} + 105 + 175\sqrt{5} + 875 + 525\sqrt{5} + 875 + 125\sqrt{5}$

$= 1856 + 832\sqrt{5}$

$(1-\sqrt{5})^7$

$= 1 - 7\sqrt{5} + 105 - 175\sqrt{5} + 875 - 525\sqrt{5} + 875 - 125\sqrt{5}$

$= 1856 - 832\sqrt{5}$

$(1856 + 832\sqrt{5}) - (1856 - 832\sqrt{5})$

$= 1856 + 832\sqrt{5} - 1856 + 832\sqrt{5}$

$= 1664\sqrt{5}$

9 Find the value of the following by choosing an appropriate value for x in the expansion of $(1+x)^n$.

a ${}^nC_0 - 2 \times {}^nC_1 + 4 \times {}^nC_2 - 8 \times {}^nC_3 + \ldots + (-1)^r 2^r \times {}^nC_r + \ldots + (-1)^n 2^n \times {}^nC_n$

b ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots {}^nC_r + \ldots + {}^nC_n$

(a) ${}^nC_r(-1)^n(2)^r$

$(1-x)^n = (1-2)^n$

$= (-1)^n$

(b) ${}^nC_r$

$(1-x)^n = (1-1)^n$

$= 0^n = 0$