Exercise 1F

In questions **1** to **9**, prove the statements by contradiction.

1. For all $n \in \mathbb{Z}$, if n^2 is odd then n is also odd.
2. $\sqrt{3}$ is irrational.
3. $\sqrt[5]{2}$ is irrational.
4. For all $p, q \in \mathbb{Z}$, $p^2 - 8q - 11 \neq 0$.
5. For all $a, b \in \mathbb{Z}$, $12a^2 - 6b^2 \neq 0$.
6. If $a, b, c \in \mathbb{Z}$, where c is an odd number and $a^2 + b^2 = c^2$, then either a or b is an even number.
7. If $n, k \in \mathbb{Z}$, then $n^2 + 2 \neq 4k$.
8. If p is an irrational number and q is a rational number, then $p + q$ is also irrational.
9. Given that m and n are positive integers, it follows that $m^2 - n^2 \neq 1$.
10. Show by a counterexample that the following statements are not true in general:
    - **a** $(m + n)^2 \neq m^2 + n^2$
    - **b** If a positive integer is divisible by a prime number, then the number is not prime.
    - **c** $2^n - 1$ is a prime number for all $n \in \mathbb{N}$.
    - **d** $2^n - 1$ is a prime number for all $n \in \mathbb{Z}^+$.
    - **e** The sum of three consecutive positive integers is always divisible by 4.
    - **f** The sum of four consecutive positive integers is always divisible by 4.

1) Assume that if n is even then n^2 is odd

$n = 2k, k \in \mathbb{Z}^+$

$\therefore n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$

$\therefore$ Hence, if n is even, then n^2 is even. This show contradiction with the initial assumption.

2) Assume $\sqrt{3} = \frac{p}{q}$; $p, q \in \mathbb{Z}, q \neq 0$, p & q can't have common factors,

$$\sqrt{3} = \frac{p}{q}$$

$$3 = \frac{p^2}{q^2}$$

$3q^2 = p^2 \Rightarrow p = 3k$; p is divisible by 3

$$3q^2 = (3k)^2$$

$q^2 = 3k^2 \Rightarrow q^2$ is divisible by 3 ; q is divisible by 3

$\therefore$ p and q are both divisible by 3. Hence they have 3 as their common factor.

This contradicts with the initial assumption. $\sqrt{3}$ is indeed irrational

3) Assume that $\sqrt[5]{2}$ is rational,

$\sqrt[5]{2} = \frac{p}{q}$; $p, q \in \mathbb{Z}$, $q \neq 0$, p and q can't have a common factor

$$\sqrt[5]{2} = \frac{p}{q}$$

$$2 = \frac{p^5}{q^5}$$

$2q^5 = p^5 \Rightarrow p = 2k$; p is divisible by 2

$$2q^5 = (2k)^5$$

$q^5 = 16k^5 \Rightarrow q = 16m$; q is divisible by 16

$\therefore$ p is divisible by 2 and q is divisible by 16. 16 and 2 have 2 as their common factor. Thus, p and q have 2 as their common factor. This contradicts the initial assumption. $\sqrt[5]{2}$ is indeed irrational.

4) Assume that $p^2 - 8q - 11 = 0$; $p, q \in \mathbb{Z}$

$p^2 - 8q - 11 = 0$

$p^2 = 8q + 11$

$= 2(4q + 5) + 1$ $\Rightarrow$ p is odd, $p = 2k + 1$; $k \in \mathbb{Z}$

$(2k+1)^2 = 8q + 11$

$4k^2 + 4k - 8q = 10$

$2k^2 + 2k - 4q = 5$

$2(k^2 + k - 2q) = 5$

$\therefore$ LHS is an even number but RHS is a odd number

contradiction happens. hence $p^2 - 8q - 11 \neq 0$

5) Assume that $12a^2 - 6b^2 = 0$; $a, b \in \mathbb{Z} \setminus [0]$

$12a^2 - 6b^2 = 0$

$12a^2 = 6b^2$

$2a^2 = b^2$

$2 = \frac{b^2}{a^2}$

$\sqrt{2} = \frac{b}{a}$

$\therefore$ Contradiction occurs. $\sqrt{2}$ is irrational.

Hence $12a^2 - 6b^2 \neq 0$

6) Assume that $a^2+b^2=c^2$ where a or b is even when c is odd,

<u>case ①</u>

$$a^2+b^2=c^2$$
$$(2p)^2+(2q+1)^2=(2r)^2$$
$$4p^2+4q^2+4q+1=4r^2$$
$$4p^2+4q^2-4r^2+4q=-1$$
$$2(2p^2+2q^2-2r^2+2q)=-1$$

<u>case ②</u>

$$a^2+b^2=c^2$$
$$(2p+1)^2+(2q+1)^2=(2r)^2$$
$$4p^2+4p+1+4q^2+4q+1=4r^2$$
$$4p^2+4p+4q^2+4q-4r^2=-2$$
$$2p^2+2p+2q^2+2q-2r^2=-1$$
$$2(p^2+p+q^2+q-r^2)=-1$$

∴ For both cases, LHS is even but RHS is odd. Contradiction.

7) Assume that $n^2+2=4k$; $n,k \in \mathbb{Z}$

$$n^2+2=4k$$
$$n^2=4k-2$$
$$n^2=2(2k-1) \Rightarrow n \text{ is even};\ n=2k$$
$$(2k)^2=2(2k-1)$$
$$2k^2=2k-1$$

∴ LHS is even but RHS is odd. Contradiction.

$$n^2+2 \neq 4k$$

8) Assume that $p+q = \frac{a}{b}$; $a,b \in \mathbb{Z}$, $p \in \mathbb{Q}$ and $q \in \bar{\mathbb{Q}}$

$$p+q = \frac{a}{b}$$

$$\frac{c}{d}+q = \frac{a}{b} \; ; \; c,d \in \mathbb{Z}$$

$$q = \frac{a}{b} - \frac{c}{d}$$

$$q = \frac{ad-cb}{bd}$$

$\therefore$ q was asumed to be irrational but we got q as rational.

Contradiction occurs.

9) Assume that $m^2-n^2=1$; $m,n \in \mathbb{Z}^+$

$$m^2-n^2=1$$

$(m-n)(m+n)=1 \Rightarrow (m-n),(m+n)$ could only be 1 or -1 at a time

case ①,	case ②,	
$m-n=-1$ $m+n=-1$	$m-n=1$ $m+n=1$	$\therefore n=0,\ n=-1$
$m=n-1$ $m=-n-1$	$m=n+1$ $m=1-n$	contradiction
$n-1=-n-1$	$n+1=1-n$	
$2n=0$	$2n=-2$	
$n=0$	$n=-1$	

(a) $(m+n)^2 \neq m^2 + n^2$

let $m = 1, n = 0$

$LHS = (1+0)^2$	$RHS = 1^2 + 0^2$
$= 1$	$= 1$

$\therefore$ By counterexample : $m = 1$, $n = 0$

the statement is NOT TRUE

(b) positive integer = 3 ,

prime number = 3

$$\frac{3}{3} = 1$$

$\therefore$ Positive integer which is 3 is divisible by a prime number 3. But 3 (the positive integer) is a prime number too. Hence, the statement is NOT TRUE.

(c) let $n = 4$

$$2^{(4)} - 1$$

$$= 15 \quad = 3(5)$$

$\therefore$ When $n = 4$, $2^n - 1$ is 15 which is not a prime number. Hence the statement is NOT TRUE.

(d) let $n = 4$

$2^{(4)} - 1$

$= 15 \quad = 3(5)$

$\therefore$ when $n = 4$, $2^n - 1$ is 15 which is not a prime number. Hence the statement is NOT TRUE.

(e) $k + (k+1) + (k+2)$

$= 3k + 3$

let $k = 2$,

$3(2) + 3$ $\quad \therefore$ not divisible by 4

$= 9$

(f) $k + (k+1) + (k+2) + (k+3)$

$= 4k + 6$

let $k = 1$,

$4(1) + 6$ $\quad \therefore$ not divisible by 4

$= 10$