Exercise 1E

1 Prove that $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$.

2 Show that the product of two odd numbers is always an odd number.

3 Prove that a four-digit number is divisible by 9 if the sum of its digits is divisible by 9. Hence identify which of the numbers 3978, 5453, 7898, 9864, 5670 are divisible by 9 without carrying out any division.

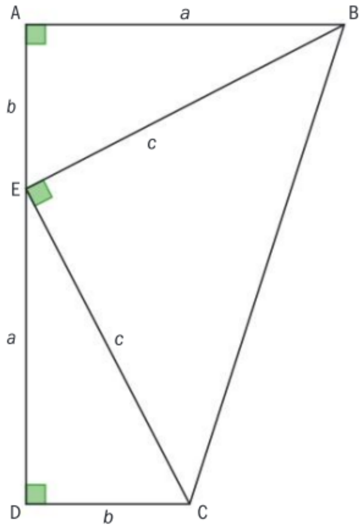
4 Show that $(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2$.

5 Prove that $\frac{1}{3} - \frac{2}{9} + \frac{1}{27} - \frac{2}{81} + \frac{1}{243} - \frac{2}{729} + \ldots = \frac{1}{8}$.

6 Prove that the difference between the squares of two consecutive numbers is always an odd number.

7 Show that $\frac{1}{(n-1)} - \frac{1}{n} + \frac{1}{(n+1)} = \frac{n^2+1}{n(n^2-1)}$. Hence determine the value of $\frac{1}{5} - \frac{1}{6} + \frac{1}{7}$.

8 The diagram here shows a trapezium ABCD that has been divided into three triangles. Use your knowledge of areas and the diagram to show that $a^2 + b^2 = c^2$.



1) $(a+b)^2 + (a-b)^2 = a^2 + 2ab + b^2 + [a^2 - 2ab + b^2]$

$= 2a^2 + 2b^2$

$= 2(a^2 + b^2)$

2) $(2n+1)(2n+1)$

$= 4n^2 + 2n + 2n + 1$

$= 4n^2 + 4n + 1$

$= 2(2n^2 + 2n) + 1$

$\therefore$ odd

3) $a_3a_2a_1a_0$

$$N = a_3 \times 10^3 + a_2 \times 10^2 + a_1 \times 10 + a_0$$

given that $a_3 + a_2 + a_1 + a_0 = 9m$, $m \in \mathbb{Z}^+$

$$N = (999+1)a_3 + (99+1)a_2 + (9+1)a_1 + a_0$$
$$= (999a_3 + 99a_2 + 9a_1) + (a_3 + a_2 + a_1 + a_0)$$
$$= 9(111a_3 + 11a_2 + a_1) + 9m$$
$$= 9(111a_3 + 11a_2 + a_1 + m)$$

4) $(ad+bc)^2 + (bd-ac)^2 = [a^2d^2 + 2abcd + b^2c^2] + [b^2d^2 - 2abcd + a^2c^2]$

$$= a^2d^2 + b^2c^2 + b^2d^2 + a^2c^2$$
$$= a^2(d^2+c^2) + b^2(c^2+d^2)$$
$$= (a^2+b^2)(c^2+d^2)$$

5) $\frac{1}{3} - \frac{2}{9} + \frac{1}{27} - \frac{2}{81} + \frac{1}{243} - \frac{2}{729} + \dots = \frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \dots - \frac{2}{9} - \frac{2}{81} - \frac{2}{729} + \dots$

$$= \frac{1}{3}\left[1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \dots\right] + \dots - 2\left[\frac{1}{9} + \frac{1}{9}\left(\frac{1}{9}\right) + \frac{1}{9}\left(\frac{1}{9}\right)^2 + \dots\right]$$
$$= \frac{1}{3}\left[\frac{1}{1-\left(\frac{1}{9}\right)}\right] - 2\left[\frac{\frac{1}{9}}{1-\left(\frac{1}{9}\right)}\right]$$
$$= \frac{1}{8}$$

(6) $n^2 + (n+1)^2$

$= n^2 + n^2 + 2n + 1$

$= 2n^2 + 2n + 1$

$= 2(n^2 + n) + 1$

$\therefore$ odd

(7) $\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1}$

$= \frac{n - n + 1}{n(n-1)} + \frac{1}{n+1}$

$= \frac{n+1 + (n^2 - n)}{n(n+1)(n-1)}$

$= \frac{n^2 + 1}{n(n^2+1)}$

$\frac{1}{5} - \frac{1}{6} + \frac{1}{7} = \frac{6^2 + 1}{6(6^2 - 7)}$

$= \frac{37}{6(29)}$

$= \frac{37}{174}$

8. Area of trapezium $= \frac{1}{2}(a+b)(a+b)$

$= \frac{1}{2}(a^2 + 2ab + b^2)$

total area of triangles $= \frac{1}{2}(a)(b) + \frac{1}{2}(c)(c) + \frac{1}{2}(a)(b)$

$= \frac{2ab + c^2}{2}$

$\frac{1}{2}(a^2 + 2ab + b^2) = \frac{2ab + c^2}{2}$

$a^2 + 2ab + b^2 = 2ab + c^2$

$a^2 + b^2 = c^2$